PERGAMON

International Journal of Solids and Structures 38 (2001) 2533–2548

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

On the interaction between an edge dislocation and a coated inclusion

Z.M. Xiao *, B.J. Chen

School of Mechanical and Production Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 639798, Singapore

Received 19 July 1999; in revised form 6 April 2000

Abstract

An analytical solution is derived for the stress field due to an edge dislocation located near a coated inclusion in a solid by using the Muskhelishvili complex variable method. The force on the dislocation is calculated. Examples for various coating thickness and material constant combinations are given and discussed. It is shown that when a coating layer is thick, the elastic properties of the inclusion have no significant influence on the force on the dislocation. Therefore, the equilibrium position and the stability of the dislocation can be obtained in a manner similar to the two-phase model adopted by Dundurs and Mura (Dundurs, J., Mura, T., 1964. Journal of Mechanics and Physics of Solids 12, 177–189.). If the coating layer is thin, both the shear modulus and Poisson's ratio of both the inclusion and the coating can affect and change greatly the equilibrium position and the stability of the dislocation. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Edge dislocation; Coating; Inclusion; Interaction

1. Introduction

The study of dislocations interacting with inhomogeneities is motivated by the need to gain a better understanding of certain strengthening and hardening materials, especially composite materials. Foreign atoms in a solid matrix, vacancies in the crystal lattice and other kinds of inclusions (such as fibers in matrix, precipitates in alloys), because of their interaction with dislocations, play an important role among all the other factors that determine the mechanical properties of materials.

The first investigation to assess the interaction of dislocations with inhomogeneities was performed by Head (1953), who considered a dislocation near an interface between two dissimilar materials. The force on the dislocation was analyzed and a simple attraction–repulsion criterion was given in the paper. For the interaction problem of a dislocation near a circular inclusion (a circular fiber) in fiber-reinforced composite materials, Dundurs and Mura (1964) indicated that an edge dislocation may have stable equilibrium positions near the inclusion. However, a screw dislocation is simply either repelled or attracted by the

* Corresponding author. Tel.: +65-790-4726; fax: +65-791-1859.
E-mail address: mzxiao@ntu.edu.sg (Z.M. Xiao).

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PII: S0020-7683(00)00169-4

inclusion (Dundurs, 1967). The importance of studying dislocation equilibrium positions is due to the fact that a pile-up of dislocations could coalesce into a crack nucleus (Zener, 1948).

The above work on interaction between dislocations and inhomogeneities involves an isolated inhomogeneity only. For two-phase materials, when the inclusion phase has a finite concentration, the dislocations interact not only with the nearest inclusion but also with the surrounding ones. In order to reflect the mean effect of these interactions, Christensen and Lo (1979) and Christensen (1979) introduced a three-phase composite cylinder model. In the two-dimensional case, the model consists of three concentric regions: The inner circular region represents the inclusion phase, the intermediate annular region represents the matrix phase and the outer infinitely extended region represents the composite phase. Luo and Chen (1991) studied the problem of an edge dislocation located in the intermediate matrix phase based on the three-phase composite cylinder model.

On the other hand, in fiber-reinforced composite materials, coating on fibers is widely employed in order to increase the bonding strength between fibers and matrix. It has now been widely recognized that considerable toughness in composites with brittle but strong reinforcing fibers can be achieved by controlled debonding of the fibers from the matrix to prevent premature fiber fracture. A coating layer around the fiber–matrix interface can help achieve controlled delamination of the interface and prevent cracks initiated external to the fiber from damaging the matrix. Some research work on the effect of coating on mechanical properties of composite materials can be found in the open literature. To name a few, an investigation of stress field due to a coated fiber embedded in infinite matrix was carried out by Mikata and Taya (1985a,b). Qiu and Weng (1991) derived the effective elastic moduli for thickly coated particles and fiber reinforced composites based on the models of Hasin (1962) and Hasin and Rosen (1964).

The present study investigates the interaction between an edge dislocation and a coated circular inclusion. A closed-form analytical solution is derived for the stress field due to an edge dislocation near a coated inclusion. The force on the dislocation is calculated and the equilibrium positions of the dislocation are discussed under various material property combinations and coating thickness.

2. Formulation

The physical problem to be studied is shown in Fig. 1, where an edge dislocation with Burgers vector $\boldsymbol{B}_{\mathrm{V}} = b_x + \mathrm{i}b_y$ is located at point $(e, 0), e > b$ near a coated fiber. Region 1, the fiber with elastic properties

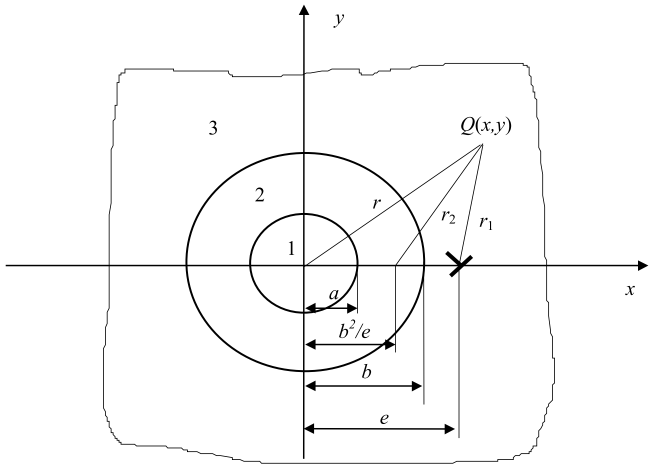

Fig. 1. An edge dislocation near a coated inclusion.

κ_1, μ_1, occupies the inner area $r \leqslant a$; region 2, the coating layer with elastic properties κ_2, μ_2, occupies the intermediate area $a \leqslant r \leqslant b$, and region 3, the matrix material with elastic properties κ_3, μ_3, occupies the outer area $r \geqslant b$. The three regions are denoted below by phases 1–3, respectively.

For a region bounded by a circle, say, $|z| = c$, its stress and displacement fields can be expressed in terms of Muskhelishvili's complex potentials as

$$\sigma_{rr} + \sigma_{\theta\theta} = 2[\phi(z) + \overline{\phi}(\bar{z})], \tag{1}$$

$$\sigma_{rr} + \mathrm{i}\sigma_{r\theta} = \phi(z) - \phi\left(\frac{c^2}{\bar{z}}\right) + \bar{z}\left(\frac{\bar{z}}{c^2} - \frac{1}{z}\right)\overline{\psi}(\bar{z}), \tag{2}$$

$$u' + \mathrm{i}v' = \frac{\mathrm{i}z}{2\mu}\left[\kappa\phi(z) + \phi\left(\frac{c^2}{\bar{z}}\right) - \bar{z}\left(\frac{\bar{z}}{c^2} - \frac{1}{z}\right)\overline{\psi}(\bar{z})\right] \tag{3}$$

in cylindrical coordinates, where $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, $u' = \partial u_x/\partial\theta$, $v' = \partial u_y/\partial\theta$, $\kappa = 3 - 4v$, v is Poisson's ratio and μ is the shear modulus. In addition, the function $\psi(z)$ can be expressed in terms of $\phi(z)$ as

$$\psi(z) = \frac{c^2}{z^2}\left[\phi(z) + \overline{\phi}\left(\frac{c^2}{z}\right) - z\phi'(z)\right]. \tag{4}$$

For the current problem, the continuity of displacement and traction across the interfaces between the three phases requires

$$\sigma_{rr}^{(1)}(\tau) + \mathrm{i}\sigma_{r\theta}^{(1)}(\tau) = \sigma_{rr}^{(2)}(\tau) + \mathrm{i}\sigma_{r\theta}^{(2)}(\tau), \tag{5}$$

$$u'_{(1)}(\tau) + \mathrm{i}v'_{(1)}(\tau) = u'_{(2)}(\tau) + \mathrm{i}v'_{(2)}(\tau), \tag{6}$$

where $\tau = ae^{\mathrm{i}\theta}$ with $0 \leqslant \theta \leqslant 2\pi$, and

$$\sigma_{rr}^{(2)}(\varsigma) + \mathrm{i}\sigma_{r\theta}^{(2)}(\varsigma) = \sigma_{rr}^{(3)}(\varsigma) + \mathrm{i}\sigma_{r\theta}^{(3)}(\varsigma), \tag{7}$$

$$u'_{(2)}(\varsigma) + \mathrm{i}v'_{(2)}(\varsigma) = u'_{(3)}(\varsigma) + \mathrm{i}v'_{(3)}(\varsigma) \tag{8}$$

in which $\varsigma = be^{\mathrm{i}\theta}$ with $0 \leqslant \theta \leqslant 2\pi$. By means of Eqs. (2) and (3), the traction continuity conditions (7) and (8) are written as

$$\phi_{(2)}^{+}(\varsigma) + \phi_{(3)}^{+}(\varsigma) = \phi_{(2)}^{-}(\varsigma) + \phi_{(3)}^{-}(\varsigma), \tag{9}$$

$$\frac{\kappa_2}{2\mu_2}\phi_{(2)}^{+}(\varsigma) + \frac{1}{2\mu_2}\phi_{(2)}^{-}(\varsigma) = \frac{\kappa_3}{2\mu_3}\phi_{(3)}^{-}(\varsigma) + \frac{1}{2\mu_3}\phi_{(3)}^{+}(\varsigma), \tag{10}$$

where the superscripts "+" and "−" denote the value of the function obtained as z approaches to $\varsigma = be^{\mathrm{i}\theta}$ from the interior and the exterior of the circle $|z| = b$, respectively. Using the continuation theory given by England (1971), and with reference to the structure of $\phi_{(2)}(z)$ and $\phi_{(3)}(z)$, the approach demonstrates that the function $\Xi_{(1)}(z) = \phi_{(2)}(z) + \phi_{(3)}(z)$ is holomorphic in the finite region $a < |z| < b^2/a$ except some singularities at poles $z = e$ and $z = b^2/e$; so we express $\Xi_{(1)}(z)$ as

$$\Xi_{(1)}(z) = \frac{\gamma}{z - e} - \frac{\gamma}{z - b^2/e} + \frac{\overline{\gamma}}{e^3}\frac{b^2(e^2 - b^2)}{(z - b^2/e)^2} + \frac{\gamma}{z}\sum_{n=-\infty}^{\infty} A_n z^n, \tag{11}$$

where the last three terms consist of a Laurent series which is convergent in the region

$$a < |z| < b^2/a \quad \text{and} \quad \gamma = \frac{\mu}{\pi(1+\kappa)}(b_y - \mathrm{i}b_x),\ \overline{\gamma} = \frac{\mu}{\pi(1+\kappa)}(b_y + \mathrm{i}b_x).$$

Since at $z = \varsigma = be^{i\theta}$, $\Xi_{(1)}(\varsigma) = \phi_{(2)}(\varsigma) + \phi_{(3)}(\varsigma)$, by virtue of Eq. (11), Eq. (10) is reduced to the Hilbert problem,

$$\phi^{+}_{(3)}(\varsigma) + \eta_1 \phi^{-}_{(3)}(\varsigma) = f_1(\varsigma) \tag{12}$$

with

$$\eta_1 = \frac{\kappa_3\mu_2 + \mu_3}{\kappa_2\mu_3 + \mu_2}, \tag{13}$$

$$f_1(\varsigma) = \beta_0\left[\frac{\gamma}{\varsigma - e} - \frac{\gamma}{\varsigma - b^2 e} + \frac{\overline{\gamma}}{e^3}\frac{b^2(e^2-b^2)}{(\varsigma - b^2/e)^2} + \frac{\gamma}{\varsigma} + \sum_{n=-\infty}^{\infty} A_n\varsigma^n\right], \tag{14}$$

$$\beta_0 = \frac{\mu_3(\kappa_2+1)}{\kappa_2\mu_3 + \mu_2} = 1 - \beta_1 + \eta_1, \quad \beta_1 = \frac{\mu_2(\kappa_3+1)}{\kappa_2\mu_3+\mu_2}, \tag{15}$$

Introducing a sectionally holomorphic auxiliary function $\Omega_1(z)$ as

$$\Omega_1(z) = \begin{cases} \phi_{(3)}(z) & \text{when } |z| < b, \\ -\eta_1\phi_{(3)}(z) & \text{when } |z| > b, \end{cases} \tag{16}$$

the Hilbert problem (12) then becomes

$$\Omega_1^{+}(\varsigma) - \Omega_1^{-}(\varsigma) = f_1(\varsigma) \tag{17}$$

for $\varsigma = be^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$. Using Plemelj's theorem (Muskhelishvili, 1953), we have

$$\Omega_1(z) = \frac{1}{2\pi i}\oint_{|\varsigma|=b}\frac{f_1(\varsigma)}{\varsigma - z}d\varsigma + \frac{\gamma}{z} - \frac{\gamma}{z - b^2/e} + \frac{\overline{\gamma}}{e^3}\frac{b^2(e^2-b^2)}{(z-b^2/e)^2} - \eta_1\frac{\gamma}{z-e}. \tag{18}$$

Substituting Eq. (14) into Eq. (18) leads to

$$\phi_{(3)}(z) = \frac{\gamma}{z} - \frac{\gamma}{z-b^2/e} + \frac{\overline{\gamma}}{e^3}\frac{b^2(e^2-b^2)}{(z-b^2/e)^2} + (1-\beta_1)\frac{\gamma}{z-e} + (1-\beta_1+\eta_1)\sum_{n=0}^{\infty}A_n z^n \tag{19a}$$

for $|z| < b$ and

$$\phi_{(3)}(z) = \frac{\gamma}{z-e} + \left(1 - \frac{\beta_1}{\eta_1}\right)\left[\frac{\gamma}{z} - \frac{\gamma}{z-b^2/e} + \frac{\overline{\gamma}}{e^3}\frac{b^2(e^2-b^2)}{(z-b^2/e)^2}\right] + \frac{1-\beta_1+\eta_1}{\eta_1}\sum_{n=1}^{\infty}A_{-n}z^{-n} \tag{19b}$$

for $|z| > b$. Subtracting $\phi_{(3)}(z)$ from $\Xi_1(z)$ leads to

$$\phi_{(2)}(z) = \beta_1\frac{\gamma}{z-e} + (\beta_1 - \eta_1)\sum_{n=0}^{\infty}A_n z^n + \sum_{n=1}^{\infty}A_{-n}z^{-n} \tag{20a}$$

for $a < |z| < b$ and

$$\phi_{(2)}(z) = \frac{\beta_1}{\eta_1}\left[\frac{\gamma}{z} - \frac{\gamma}{z-b^2/e} + \frac{\overline{\gamma}}{e^3}\frac{b^2(e^2-b^2)}{(z-b^2/e)^2}\right] + \frac{\beta_1 - 1}{\eta_1}\sum_{n=1}^{\infty}A_{-n}z^{-n} + \sum_{n=0}^{\infty}A_n z^n \tag{20b}$$

for $b < |z| < b^2/a$. Then, substitution of Eqs. (19) and (20) into Eq. (4) with $c = b$ leads to

$$\psi_{(2)}(z) = \frac{b^2}{z^2}\Bigg\{\overline{A}_0 + (\beta_1 - \eta_1)A_0 + \frac{\beta_1}{\eta_1}\frac{e}{b^2}(\overline{\gamma} + \gamma) - \frac{\beta_1}{\eta_1}\frac{\gamma}{e} + \frac{\beta_1}{\eta_1}\frac{\overline{\gamma}}{b^2}z + \left[2\beta_1\left(1 - \frac{1}{\eta_1}\right)\gamma + \frac{\beta_1}{\eta_1}\frac{e^2}{b^2}(2\gamma + \overline{\gamma})\right]\frac{1}{z - e} + \left[\beta_1\left(1 - \frac{1}{\eta_1}\right)e\gamma + \frac{\beta_1}{\eta_1}\frac{e^3}{b^2}\gamma\right]\frac{1}{(z - e)^2} - (\beta_1 - \eta_1)\sum_{n=1}^{\infty}A_n(n-1)z^n - \frac{1-\beta_1}{\eta_1}\sum_{n=1}^{\infty}\overline{A}_{-n}\left(\frac{z}{b^2}\right)^n + \sum_{n=1}^{\infty}A_{-n}(n+1)z^{-n} + \sum_{n=1}^{\infty}\overline{A}_n\left(\frac{b^2}{z}\right)^n\Bigg\} \tag{21}$$

for $a < |z| < b$ and

$$\psi_{(3)}(z) = \frac{b^2}{z^2}\Bigg\{(1 - \beta_1 + \eta_1)\overline{A}_0 + \frac{(e^2 - b^2)\gamma + e^2\overline{\gamma}}{eb^2} - (1 - \beta_1)\frac{\overline{\gamma}}{e} + \frac{e^2}{b^2}(2\gamma + \overline{\gamma})\frac{1}{z - e} + \frac{e^3}{b^2}\gamma\frac{1}{(z - e)^2} - \left[2\left(1 - \frac{\beta_1}{\eta_1}\right)\gamma + (1 - \beta_1)\frac{b^2}{e^2}\overline{\gamma}\right]\frac{1}{z - b^2/e} + \left(1 - \frac{\beta_1}{\eta_1}\right)\frac{3b^2(e^2 - b^2)\overline{\gamma} - b^2e^2\gamma}{e^3} \times \frac{1}{(z - b^2/e)^2} + \left(1 - \frac{\beta_1}{\eta_1}\right)\frac{2b^4(e^2 - b^2)\overline{\gamma}}{e^4}\frac{1}{(z - b^2/e)^3} + \left(1 - \frac{\beta_1}{\eta_1}\right)\frac{2\gamma}{z} + \frac{\overline{\gamma}}{b^2}z + \frac{1 - \beta_1 + \eta_1}{\eta_1}\sum_{n=1}^{\infty}A_{-n}(n+1)z^{-n} + (1 - \beta_1 + \eta_1)\sum_{n=1}^{\infty}\overline{A}_n\left(\frac{b^2}{z}\right)^n\Bigg\} \tag{22}$$

for $|z| > b$.

In the same manner, starting from the continuity conditions (5) and (6), we obtain the second Hilbert problem,

$$\phi^{+}_{(1)}(\tau) + \eta_2\phi^{-}_{(1)}(\tau) = f_2(\tau) \tag{23}$$

for $\varsigma = ae^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, where

$$\eta_2 = \frac{\kappa_2\mu_1 + \mu_2}{\kappa_1\mu_2 + \mu_1}, \tag{24}$$

$$f_2(\tau) = \beta_2\left(-\overline{\phi}_{(1)}(0) + \sum_{n=-\infty}^{\infty}B_n\tau^n\right) \tag{25}$$

with

$$\beta_2 = \frac{\mu_1(\kappa_2 + 1)}{\kappa_1\mu_2 + \mu_1}, \tag{26}$$

and B_ns are the coefficients of a Laurent series, which is convergent in the region $a^2/b < |z| < b$. The solution of this problem is

$$\phi_{(1)}(z) = (\eta_2 - \beta_2)\overline{\phi}_{(1)}(0) + \beta_2\sum_{n=0}^{\infty}B_nz^n \tag{27a}$$

for $|z| < a$ and

$$\phi_{(1)}(z) = -\overline{\phi}_{(1)}(0) + \frac{\beta_2}{\eta_2}\sum_{n=1}^{\infty} B_{-n} z^{-n} \tag{27b}$$

for $|z| > a$. From Eq. (27a), we get two equations:

$$\phi_{(1)}(0) - (\eta_2 - \beta_2)\overline{\phi}_{(1)}(0) = \beta_2 B_0, \qquad (\eta_2 - \beta_2)\phi_{(1)}(0) - \overline{\phi}_{(1)}(0) = -\beta_2 \overline{B}_0. \tag{28}$$

So $\overline{\phi}_{(1)}(0)$ and $\phi_{(1)}(0)$ can be determined as

$$\overline{\phi}_{(1)}(0) = \frac{\beta_2 \overline{B}_0 + \beta_2(\eta_2 - \beta_2)B_0}{1 - (\eta_2 - \beta_2)^2}, \qquad \phi_{(1)}(0) = \frac{\beta_2 B_0 + \beta_2(\eta_2 - \beta_2)\overline{B}_0}{1 - (\eta_2 - \beta_2)^2}. \tag{29}$$

Subtracting $\phi_{(1)}(z)$ from $\Xi_2(z)$, we obtain

$$\phi_{(2)}(z) = -(1 + \eta_2 - \beta_2)\overline{\phi}_{(1)}(0) + \sum_{n=1}^{\infty} B_{-n} z^{-n} + (1 - \beta_2)\sum_{n=0}^{\infty} B_n z^n \tag{30a}$$

for $a^2/b < |z| < a$ and

$$\phi_{(2)}(z) = \left(1 - \frac{\beta_2}{\eta_2}\right)\sum_{n=1}^{\infty} B_{-n} z^{-n} + \sum_{n=0}^{\infty} B_n z^n \tag{30b}$$

for $a < |z| < b$. Finally, substitution of Eqs. (29) and (30) into Eq. (4) with $c = a$ leads to

$$\psi_{(2)}(z) = \frac{a^2}{z^2}\left\{ \frac{1 - \eta_2}{1 + \beta_2 - \eta_2}(B_0 + \overline{B}_0) - \sum_{n=1}^{\infty}(n-1)B_n z^n + \sum_{n=1}^{\infty}\overline{B}_{-n}\left(\frac{z}{a^2}\right)^n + \left(1 - \frac{\beta_2}{\eta_2}\right)\sum_{n=1}^{\infty}(n+1)B_{-n} z^{-n} + (1 - \beta_2)\sum_{n=1}^{\infty}\overline{B}_n\left(\frac{a^2}{z}\right)^n \right\} \tag{31}$$

for $a < |z| < b$ and

$$\psi_{(1)}(z) = \frac{a^2}{z^2}\left\{ \frac{\beta_2}{\eta_2}\sum_{n=1}^{\infty}\overline{B}_{-n}\left(\frac{z}{a^2}\right)^n - \beta_2\sum_{n=1}^{\infty}(n-1)B_n z^n \right\} \tag{32}$$

for$|z| < a$.

In order to satisfy the continuity conditions of displacement and traction at the interfaces $|z| = a$ and $|z| = b$ simultaneously, the function $\phi_{(2)}(z)$ expressed in Eqs. (20a) and (30b) and $\psi_{(2)}(z)$ in Eqs. (20) and (31) must be compatible, respectively (England, 1971). Thus, we obtain two compatibility identities. Expanding both sides of each identity into power series of z/a and comparing the coefficients of their like powers, we obtain a set of simultaneous equations with respect to unknowns $A_n^* = A_n a^n$, $B_n^* = B_n a^n (n = 0, \pm 1, \pm 2, \ldots)$. If we further define

$$A_n^* = \frac{\mu_2}{\pi(1 + \kappa_2)b}(a_n b_y + \mathrm{i} a_n' b_x), \qquad B_n^* = \frac{\mu_2}{\pi(1 + \kappa_2)b}(b_n b_y + \mathrm{i} b_n' b_x), \tag{33}$$

we get two sets of real simultaneous equations to determine a_n, a_n', b_n and b_n':

$$\frac{(1 - D)C}{1 - C} a_n + b_n = -(1 - D)\left(\frac{1}{\beta}\right)^{n+1} \quad (n \geqslant 0), \tag{34a}$$

$$(1 - D)a_{-n} - A b_{-n} = 0 \quad (n \geqslant 1), \tag{34b}$$

$$-\frac{(1-D)C}{1-C}(n-1)a_n+\left[\frac{(1-C)D}{1-D}-\frac{1-CD}{1-D}\delta_{n0}\right]a_{-n}-[n-1+(1-M)\delta_{n0}]\alpha^2 b_n$$
$$+[1-(1-M)\delta_{n0}]\alpha^{-2n+2}b_{-n}=(1-C)\frac{1}{\beta}(2\beta^2-1)\delta_{n0}+(1-C)\delta_{n1}$$
$$+(n-1)(C-D)\left(\frac{1}{\beta}\right)^{n+1}+(n-2)(1-C)\left(\frac{1}{\beta}\right)^{n-1}\quad(n\geqslant 0), \tag{34c}$$

$$(n+1)a_{-n}+a_n-A(n+1)\alpha^2 b_{-n}-B\alpha^{2n+2}b_n=0\quad(n\geqslant 1), \tag{34d}$$

$$\frac{(1-D)C}{1-C}a'_n+b'_n=(1-D)\left(\frac{1}{\beta}\right)^{n+1}\quad(n\geqslant 0), \tag{35a}$$

$$(1-D)a'_{-n}-Ab'_{-n}=0\quad(n\geqslant 1), \tag{35b}$$

$$-\frac{(1-D)C}{1-C}(n-1)a'_n-\left[\frac{(1-C)D}{1-D}-\frac{1-CD}{1-D}\delta_{n0}\right]a'_{-n}-[n-1+(1-M)\delta_{n0}]\alpha^2 b'_n$$
$$-[1-(1-M)\delta_{n0}]\alpha^{-2n+2}b'_{-n}=\frac{1}{\beta}(1-C)\delta_{n0}+(1-C)\delta_{n1}$$
$$-\left[(n-1)(C-D)+n(1-C)\beta^2\right]\left(\frac{1}{\beta}\right)^{n+1}\quad(n\geqslant 0) \tag{35c}$$

$$(n+1)a'_{-n}-a'_n-A(n+1)\alpha^2 b'_{-n}+B\alpha^{2n+2}b'_n=0\quad(n\geqslant 1), \tag{35d}$$

where δ_{nl} is the Kronecker delta, and

$$\begin{aligned}&A=1-\frac{\beta_2}{\eta_2},\quad B=1-\beta_2,\quad M=\frac{1-\eta_2}{1+\beta_2-\eta_2},\quad C=1-\frac{\beta_1}{\eta_1},\\&D=1-\beta_1,\quad N=\frac{1-\eta_1}{1+\beta_1-\eta_1},\quad \beta=\frac{e}{b},\quad \alpha=\frac{a}{b}.\end{aligned} \tag{36}$$

3. Stress distribution

The stress fields are related to the complex variables through

$$\begin{aligned}\sigma_{xx}&=\mathrm{Re}[2\phi(z)-\bar{z}\phi'(z)-\psi(z)],\\\sigma_{yy}&=\mathrm{Re}[2\phi(z)+\bar{z}\phi'(z)+\psi(z)],\\\sigma_{xy}&=\mathrm{Im}[\bar{z}\phi'(z)+\psi(z)].\end{aligned} \tag{37}$$

From the above stress functions, the stress fields in each of the three regions are evaluated through

$$\sigma^{(i)}_{xx}=\frac{\mu_2}{\pi(1+\kappa_2)}\left[h^{(i)}_{xxx}b_x+h^{(i)}_{xxy}b_y\right], \tag{38a}$$

$$\sigma^{(i)}_{yy}=\frac{\mu_2}{\pi(1+\kappa_2)}\left[h^{(i)}_{yyx}b_x+h^{(i)}_{yyy}b_y\right], \tag{38b}$$

$$\sigma_{xy}^{(i)} = \frac{\mu_2}{\pi(1+\kappa_2)}\left[h_{xyx}^{(i)} b_x + h_{xyy}^{(i)} b_y\right], \tag{38c}$$

where $i = 1, 2, 3$ for regions 1, 2, 3, respectively. In the current problem, of particular interest is the stress field in region 3 as the dislocation is located in this region. The detailed expressions of the influence functions for $i = 3$ are listed in Appendix A.

It is noticed that if the materials of region 1 and region 2 are the same, then $A = B = 0$. From Eqs. (34) and (35), we get

$$a_n = a'_n = a_{-n} = a'_{-n} = 0 \quad (n > 0), \tag{39}$$

and

$$a_0 = \frac{\beta_1}{\beta(1+\beta_1-\eta_1)} = \frac{(1-C)(D+1-2N)}{\beta(1-CD)} = \frac{1-N}{\beta}, \tag{40a}$$

$$a'_0 = \frac{\beta_1}{\beta(1-\beta_1+\eta_1)} = \frac{(1-D)(1-C)}{\beta(1-CD)} = \frac{N(1-D)}{\beta(2N+D-1)}. \tag{40b}$$

It is seen that the stress field is fully reduced to that derived by Dundurs and Mura (1964) for an infinite body containing a circular inclusion with an edge dislocation nearby.

4. Force on dislocation

Following Dundurs and Mura (1964), we can calculate the strain energy of the present model by evaluating the work required to introduce a dislocation into the given location. Thus, for the dislocation with its Burgers vector in the x direction, the strain energy per unit thickness can be written as

$$W_1 = \frac{1}{2} b_x \int_{e+r_0}^{R} \sigma_{xy}(x,0)\,\mathrm{d}x, \tag{41}$$

where r_0 is the radius of the core of the dislocation, and R is the outer radius of the body. It is relevant to assume $r_0 \to 0$ and $R \to \infty$. Substituting Eq. (38) into Eq. (41) with $i=3$ (refer to Appendix A), we have

$$W_1 = \frac{\mu_2 b_x^2}{2\pi(1+\kappa_2)}\left[2\log\frac{R}{r_0} + (C+D)\log\frac{\beta^2-1}{\beta^2} + \frac{1-3C}{\beta^2} + \frac{1-D}{\beta^2} - \frac{1-CD}{1-D}a'_{-1}\log\frac{R}{e} \right.$$
$$\left. - \frac{1-CD}{1-D}\sum_{n=2}^{\infty}\frac{n}{n-1}\frac{a'_{-n}}{\beta^{n-1}} + \frac{1-CD}{1-D}\sum_{n=1}^{\infty}\frac{a'_{-n}}{\beta^{n+1}} - \frac{1-CD}{1-C}\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{a'_n}{\beta^{n+1}}\right]. \tag{42}$$

Similarly, for the dislocation with its Burgers vector in the y direction, the strain energy per unit thickness can be written as

$$W_2 = \frac{1}{2} b_y \int_{e+r_0}^{R} \sigma_{yy}(x,0)\,\mathrm{d}x. \tag{43}$$

Substituting Eq. (38) into Eq. (43) with $i=3$ (refer to Appendix A) leads to

$$W_2 = \frac{\mu_2 b_x^2}{2\pi(1+\kappa_2)} \left[2\log\frac{R}{r_0} + (C+D)\log\frac{\beta^2-1}{\beta^2} - \frac{1-C}{\beta^2} + \frac{D+1-2N}{\beta^2} + \frac{1-CD}{1-D} a_{-1} \log\frac{R}{e} \right.$$
$$\left. + \frac{1-CD}{1-D}\sum_{n=2}^{\infty}\frac{n}{n-1}\frac{a_{-n}}{\beta^{n-1}} + \frac{1-CD}{1-D}\sum_{n=1}^{\infty}\frac{a_{-n}}{\beta^{n+1}} + \frac{1-CD}{1-C}\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{a_n}{\beta^{n+1}} \right]. \tag{44}$$

Assuming that the dislocation can move only in a radial direction, the force on dislocation b_x is defined as the decrease of the strain energy in the solid due to the dislocation gliding a unit distance, and the force on dislocation b_y is defined as the decrease of the strain energy in the solid due to the dislocation climbing a unit distance (Dundurs and Mura, 1964):

$$F_1 = -\frac{\partial W_1}{\partial e} = -\frac{1}{b}\frac{\partial W_1}{\partial \beta}, \qquad F_2 = -\frac{\partial W_2}{\partial e} = -\frac{1}{b}\frac{\partial W_2}{\partial \beta}, \tag{45}$$

where F_1 is the force on the glide dislocation b_x, and F_2 is the force on the climb dislocation b_y. Substituting Eqs. (42) and (44) into Eq. (45), we have

$$F_1 = -\frac{\mu_2 b_x^2}{\pi(1+\kappa_2)b} \left[\frac{C+D}{\beta(\beta^2-1)} + \frac{3C-D}{\beta^3} + \frac{1-CD}{1-D}\sum_{n=1}^{\infty} n\frac{a'_{-n}}{\beta^n} - \frac{1-CD}{1-D}\sum_{n=1}^{\infty}(n+1)\frac{a'_{-n}}{\beta^{n+2}} \right.$$
$$+ \frac{1-CD}{1-C}\sum_{n=1}^{\infty}\frac{a'_n}{\beta^{n+2}} - \frac{1-CD}{1-D}\sum_{n=2}^{\infty}\frac{n}{n-1}\frac{1}{\beta^{n-1}}\frac{\partial a'_{-n}}{\partial\beta} + \frac{1-CD}{1-D}\sum_{n=1}^{\infty}\frac{1}{\beta^{n+1}}\frac{\partial a'_{-n}}{\partial\beta}$$
$$\left. - \frac{1-CD}{1-C}\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{1}{\beta^{n+1}}\frac{\partial a'_n}{\partial\beta} \right], \tag{46}$$

$$F_2 = -\frac{\mu_2 b_y^2}{\pi(1+\kappa_2)b} \left[\frac{C+D}{\beta(\beta^2-1)} + \frac{2N-C-D}{\beta^3} - \frac{1-CD}{1-D}\sum_{n=1}^{\infty} n\frac{a_{-n}}{\beta^n} - \frac{1-CD}{1-D}\sum_{n=1}^{\infty}(n+1)\frac{a_{-n}}{\beta^{n+2}} \right.$$
$$- \frac{1-CD}{1-C}\sum_{n=1}^{\infty}\frac{a_n}{\beta^{n+2}} + \frac{1-CD}{1-D}\sum_{n=2}^{\infty}\frac{n}{n-1}\frac{1}{\beta^{n-1}}\frac{\partial a_{-n}}{\partial\beta} + \frac{1-CD}{1-D}\sum_{n=1}^{\infty}\frac{1}{\beta^{n+1}}\frac{\partial a_{-n}}{\partial\beta}$$
$$\left. + \frac{1-CD}{1-C}\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{1}{\beta^{n+1}}\frac{\partial a_n}{\partial\beta} \right], \tag{47}$$

where $\partial a_n/\partial\beta$, $\partial a_{-n}/\partial\beta$, $\partial a'_n/\partial\beta$, $\partial a'_{-n}/\partial\beta$ can be obtained directly from Eqs. (34) and (35). For the convenience of further discussion in the following sections, we define the normalized force on the dislocation as

$$F_1^* = \frac{\pi(1+\kappa_2)b}{\mu_2 b_x^2}F_1, \qquad F_2^* = \frac{\pi(1+\kappa_2)b}{\mu_2 b_y^2}F_2, \tag{48}$$

where F_1 is the force on the dislocation b_x given by Eq. (46), and F_2 is the force on the dislocation b_y given by Eq. (47).

5. Examples and discussions

In the present physical problem, the force on the dislocation is a complicated function of the material constants of three phases, the location of the dislocation and the relative thickness of the coating layer. It would make the paper too tedious and lengthy if we discussed the influence of all the parameters on the equilibrium positions of the dislocation. In the foregoing examples, our concern concentrates on how the thickness and the mechanical properties of the coating layer (phase 2) affect the equilibrium position of the dislocation.

5.1. The thickness effect of the coating layer

As a starting point, we examine the thickness effect of the coating layer on the dislocation. The normalized force F_1^* on dislocation b_x versus $\beta = e/b$ is depicted in Fig. 2 with different b/a for $\mu_1/\mu_3 = 5$, $\mu_2/\mu_3 = 1.2$, $\nu_1 = \nu_2 = 0.1$, $\nu_3 = 0.3$ for the case $\mu_1/\mu_3 > 1$ and $\mu_2/\mu_3 > 1$. It is seen that when b/a is large (the fiber is thickly coated), there is also an equilibrium position between $\beta = 1.0$–1.2. Since now $\partial F_1^*/\partial\beta > 0$, or $\partial^2 W_1/\partial\beta^2 < 0$, it is an unstable equilibrium position. However, when b/a is small, there are no equilibrium positions. The force F_2^* (the normalized force on dislocation b_y versus $\beta = e/b$ is depicted in

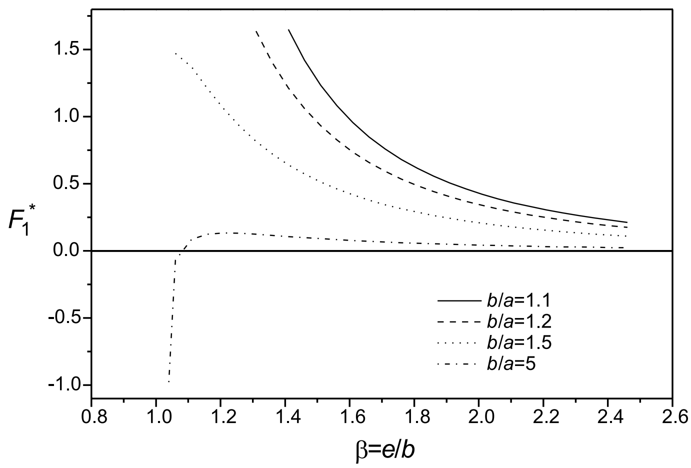

Fig. 2. Normalized force on gliding dislocation F_1^* vs. e/b for $\mu_1/\mu_3 = 5$, $\mu_2/\mu_3 = 1.2$, $\nu_1 = 0.1$, $\nu_2 = 0.1$ and $\nu_3 = 0.3$.

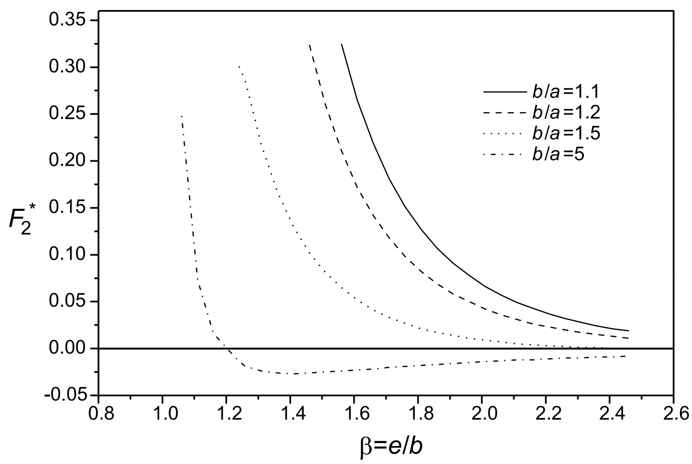

Fig. 3. Normalized force on climbing dislocation F_2^* vs. e/b for $\mu_1/\mu_3 = 5$, $\mu_2/\mu_3 = 1.2$, $\nu_1 = 0.2$, $\nu_2 = 0.2$ and $\nu_3 = 0.3$.

Fig. 3 for $\mu_1/\mu_3 = 5$, $\mu_2/\mu_3 = 1.2$, $v_1 = v_2 = 0.2$ and $v_3 = 0.3$. Similarly, When b/a is large, there is one equilibrium position, and since now $\partial F_2^*/\partial\beta < 0$, or $\partial^2 W_2/\partial\beta^2 > 0$, it is a stable equilibrium position. But there is no equilibrium position when b/a is small. From these figures, we conclude that the thickness of the coating layer can change the equilibrium position of the dislocation.

The above numerical calculations show that when the fiber is thickly coated, the results are very similar to those given in the two-phase case (Dundurs and Mura, 1974). In fact, when b/a is large, the first two terms in Eqs. (46) and (47) dominate, i.e., the influence of the fiber (phase 1) can be neglected. So, we can use the following approximation to roughly estimate the force on dislocation at this situation:

$$F_1 = -\frac{\mu_2 b_x^2}{\pi(1+\kappa_2)b}\left[\frac{C+D}{\beta(\beta^2-1)} + \frac{3C-D}{\beta^3}\right], \tag{49}$$

$$F_2 = -\frac{\mu_2 b_y^2}{\pi(1+\kappa_2)b}\left[\frac{C+D}{\beta(\beta^2-1)} + \frac{2N-C-D}{\beta^3}\right]. \tag{50}$$

Based on Eq. (49), the equilibrium position of the dislocation b_x is characterized by

$$\beta^2 = 1 - \frac{C+D}{4C}, \tag{51}$$

which is obtained by making F_1 vanish. Since $\beta > 1$, we must have either

$$C + D < 0, \quad C > 0 \tag{52}$$

or

$$C + D > 0, \quad C < 0. \tag{53}$$

It is readily shown from $\partial^2 W_1/\partial\beta^2$ that Eq. (52) is a stable equilibrium and Eq. (53) is an unstable equilibrium. It is worth noting that the material constant combinations for Fig. 2 are deliberately selected to satisfy the conditions in Eq. (53).

Similarly, based on Eq. (50), the equilibrium position of the dislocation b_y is characterized by

$$\beta^2 = 1 - \frac{C+D}{2N}, \tag{54}$$

which is obtained by making F_2 vanish. As $\beta > 1$, we must have

$$C + D < 0, \quad N > 0 \tag{55}$$

or

$$C + D > 0, \quad N < 0. \tag{56}$$

Also, it is readily shown from $\partial^2 W_2/\partial\beta^2$ that Eq. (55) is a stable equilibrium and Eq. (56) is an unstable equilibrium. Again, the material constant combinations for Fig. 3 are deliberately selected to satisfy the conditions in Eq. (55).

5.2. The effect of the shear modulus of the coating layer

Consider the following four combinations of material properties:

$$\begin{aligned} &(a) \quad \mu_1/\mu_3 = 1.2, \quad v_1 = 0.1, \quad v_2 = 0.1, \quad v_3 = 0.3, \quad b/a = 1.05, \\ &(b) \quad \mu_1/\mu_3 = 1.2, \quad v_1 = 0.2, \quad v_2 = 0.2, \quad v_3 = 0.3, \quad b/a = 1.05. \end{aligned} \tag{57}$$

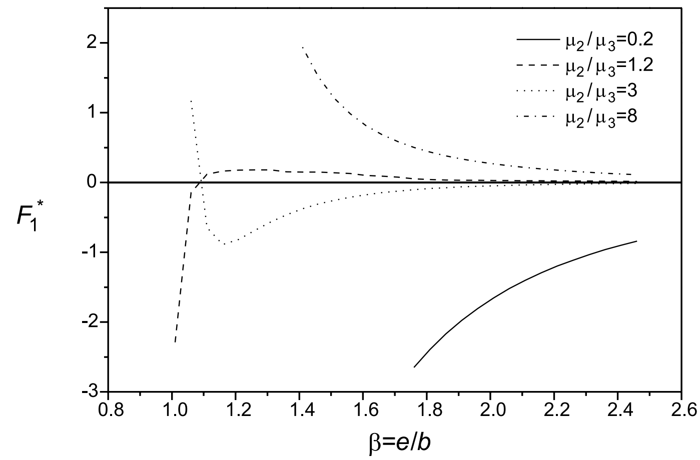

Fig. 4. Normalized force on gliding dislocation F_1^* vs. e/b for $\mu_1/\mu_3 = 1.2,\ \nu_1 = 0.1,\ b/a = 1.05,\ \nu_2 = 0.1$ and $\nu_3 = 0.3$.

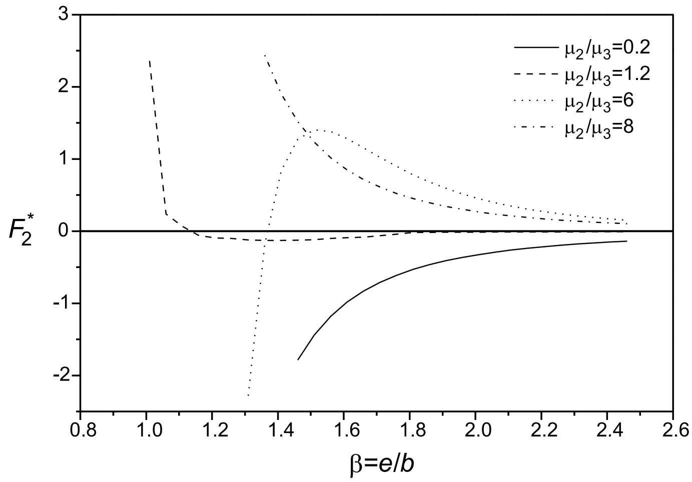

Fig. 5. Normalized force on climbing dislocation F_2^* vs. e/b for $\mu_1/\mu_3 = 1.2,\ \nu_1 = 0.2,\ b/a = 1.05,\ \nu_2 = 0.2$ and $\nu_3 = 0.3$.

Figs. 4 and 5 illustrate the variation of F_1^* (normalized force on gliding dislocation) and F_2^* (normalized force on climbing dislocation) with respect to the parameter $\beta = e/b$ for the two particular material properties with different value of μ_2/μ_3 selected above. An interesting result from Fig. 4 is that, when $\mu_2/\mu_3 = 1.2$ ($\mu_1 = \mu_2$, no coating layer exists), there is an equilibrium position between $\beta = 1.05$–1.15, where $\partial F_2^*/\partial\beta > 0$, or $\partial^2 W_2/\partial\beta^2 < 0$: therefore, it is an unstable equilibrium position. Parallel results can be obtained in Fig. 5 for normalized force F_2^* on climbing dislocation.

From Figs. 4 and 5, we observe that the shear modulus of coating layer can change not only the location of the equilibrium position, but also the stability of the equilibrium.

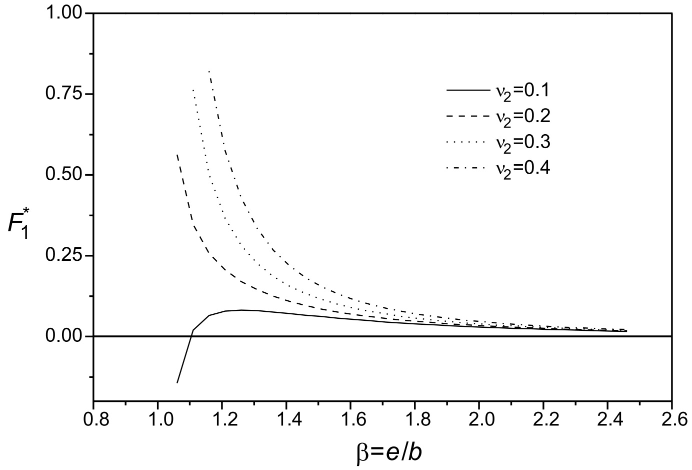

Fig. 6. Normalized force on gliding dislocation F_1^* vs. e/b for $\mu_1/\mu_3 = \mu_2/\mu_3 = 1.2,\ \nu_1 = 0.1,\ b/a = 1.1$ and $\nu_3 = 0.3$.

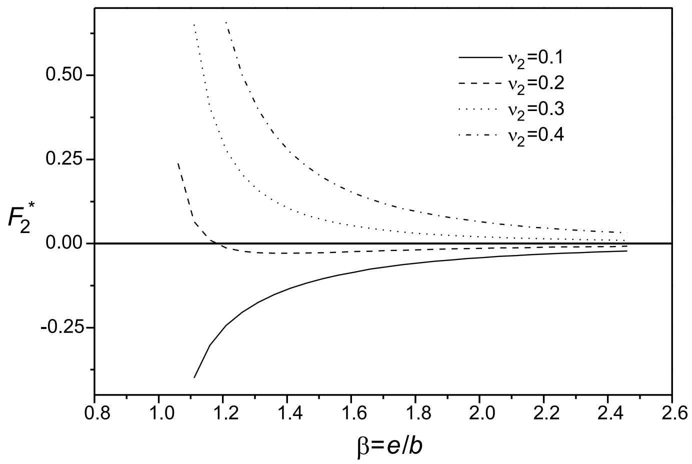

Fig. 7. Normalized force on climbing dislocation F_2^* vs. e/b for $\mu_1/\mu_3 = \mu_2/\mu_3 = 1.2,\ \nu_1 = 0.2,\ b/a = 1.1$ and $\nu_3 = 0.3$.

5.3. The effect of Poisson's ratio of the coating layer

Again, we consider the four combinations of material properties given as

$$\begin{aligned} &(\mathrm{a}) \quad \mu_1/\mu_3 = \mu_2/\mu_3 = 1.2, \quad \nu_1 = 0.1, \quad \nu_3 = 0.3, \quad b/a = 1.1, \\ &(\mathrm{b}) \quad \mu_1/\mu_3 = \mu_2/\mu_3 = 1.2, \quad \nu_1 = 0.2, \quad \nu_3 = 0.3, \quad b/a = 1.1. \end{aligned} \tag{58}$$

This time we change Poisson's ratio instead of shear modulus of the coating layer. The forces F_1^* and F_2^* versus $\beta = e/b$ for different value of ν_2 are plotted in Figs. 6 and 7, respectively. The equilibrium positions of the dislocation are obtained by letting the force on dislocation zero. It is shown from the figures that Poisson's ratio of the coating layer (phase 2) can also change the location and the stability of the equilibrium. Conclusions parallel to the previous subsection can be drawn through the figures.

Appendix A

The detailed expressions of the influence functions in Eq. (38) for $i=3$ are given as

$$h_{xxx}^{(3)} = -2\left(1+\frac{2x_1^2}{r_1^2}\right)\frac{y}{r_1^2} + \left[D + C\left(1+\frac{4x_2^2}{r_2^2}\right)\right]\frac{y}{r_2^2} - \left[D + C\left(1+\frac{4x^2}{r^2}\right)\right]\frac{y}{r^2} - 2C\left(1-\frac{4x^2}{r^2}\right)\frac{b^2y}{r^4}$$
$$-\left[\frac{1-C}{\beta} - \frac{1-CD}{1-C}a_0'\right]\frac{2bxy}{r^4} - C\frac{\beta^2-1}{\beta^3}\left[\left(1-\frac{4x_2^2}{r_2^2}\right)\frac{2x_2}{r_2^2} - \frac{\beta^2-1}{\beta}\left(1-\frac{4x_2^2}{r_2^2}\right)\frac{b}{r_2^2}\right]\frac{2by}{r_2^2}$$
$$+\frac{(1-CD)}{(1-D)b}\left\{\sum_{n=1}^{\infty}\left[\sin n\theta a'_{-n}\left(\frac{b}{r}\right)^n\right] + \sum_{n=1}^{\infty}\left[n\sin(n+2)\theta a'_{-n}\left(\frac{b}{r}\right)^n\right]\right.$$
$$\left.-\sum_{n=1}^{\infty}\left[(n+1)\sin(n+2)\theta a'_{-n}\left(\frac{b}{r}\right)^{n+2}\right]\right\} + \frac{(1-CD)}{(1-C)b}\sum_{n=1}^{\infty}\left[\sin(n+2)\theta a'_n\left(\frac{b}{r}\right)^{n+2}\right], \tag{A.1}$$

$$h_{xxy}^{(3)} = -2\left(1-\frac{2x_1^2}{r_1^2}\right)\frac{x_1}{r_1^2} + \left[D + C\left(1-\frac{4x_2^2}{r_2^2}\right)\right]\frac{x_2}{r_2^2} - \left[D + C\left(1-\frac{4x^2}{r^2}\right)\right]\frac{x}{r^2} + \frac{1}{\beta}\left(1-\frac{2x^2}{r^2}\right)$$
$$\times\left[2C\beta^2 + \beta\frac{1-CD}{1-C}a_0 - C - 1\right]\frac{b}{r^2} + 2C\left(3-\frac{4x^2}{r^2}\right)\frac{b^2x}{r^4} - C\frac{\beta^2-1}{\beta^3}\left[\beta^2\left(1-\frac{2x_2^2}{r_2^2}\right)\right.$$
$$\left.+2\left(3-\frac{4x_2^2}{r_2^2}\right)\frac{x_2^2}{r_2^2} - \frac{\beta^2-1}{\beta}\left(3-\frac{4x_2^2}{r_2^2}\right)\frac{bx_2}{r_2^2}\right]\frac{2b}{r_2^2} + \frac{(1-CD)}{(1-D)b}\left\{\sum_{n=1}^{\infty}\left[2\cos n\theta a_{-n}\left(\frac{b}{r}\right)^n\right]\right.$$
$$\left.+\sum_{n=1}^{\infty}\left[n\cos(n+2)\theta a_{-n}\left(\frac{b}{r}\right)^n\right] - \sum_{n=1}^{\infty}\left[(n+1)\cos(n+2)\theta a_{-n}\left(\frac{b}{r}\right)^{n+2}\right]\right\}$$
$$-\frac{(1-CD)}{(1-C)b}\sum_{n=1}^{\infty}\left[\cos(n+2)\theta a_n\left(\frac{b}{r}\right)^{n+2}\right], \tag{A.2}$$

$$h_{yyx}^{(3)} = -2\left(1-\frac{2x_1^2}{r_1^2}\right)\frac{y}{r_1^2} + \left[3C - D - 4C\frac{x_2^2}{r_2^2}\right]\frac{y}{r_2^2} - \left[3C - D - 4C\frac{x^2}{r^2}\right]\frac{y}{r^2} + \left[\frac{1-C}{\beta} - \frac{1-CD}{1-C}a_0'\right]$$
$$\times\frac{2bxy}{r^4} + 2C\left(1-\frac{4x^2}{r^2}\right)\frac{b^2y}{r^4} - C\frac{\beta^2-1}{\beta^3}\left[\left(-3+\frac{4x_2^2}{r_2^2}\right)\frac{2x_2}{r_2^2} + \frac{\beta^2-1}{\beta}\left(1-\frac{4x_2^2}{r_2^2}\right)\frac{b}{r_2^2}\right]$$
$$\times\frac{2by}{r_2^2} + \frac{(1-CD)}{(1-D)b}\left\{\sum_{n=1}^{\infty}\left[2\sin n\theta a'_{-n}\left(\frac{b}{r}\right)^n\right] - \sum_{n=1}^{\infty}\left[n\sin(n+2)\theta a'_{-n}\left(\frac{b}{r}\right)^n\right]\right.$$
$$\left.+\sum_{n=1}^{\infty}\left[(n+1)\sin(n+2)\theta a'_{-n}\left(\frac{b}{r}\right)^{n+2}\right]\right\} - \frac{(1-CD)}{(1-C)b}\sum_{n=1}^{\infty}\left[\sin(n+2)\theta a'_n\left(\frac{b}{r}\right)^{n+2}\right], \tag{A.3}$$

$$h_{yyy}^{(3)} = 2\left(3 - \frac{2x_1^2}{r_1^2}\right)\frac{x_1}{r_1^2} - \left[5C + D - 4C\frac{x_2^2}{r_2^2}\right]\frac{x_2}{r_2^2} + \left[5C + D - 4C\frac{x^2}{r^2}\right]\frac{x}{r^2} - \frac{1}{\beta}\left(1 - \frac{2x^2}{r^2}\right)$$
$$\times\left[2C\beta^2 + \beta\frac{1-CD}{1-C}a_0 - C - 1\right]\frac{b}{r^2} - 2C\left(3 - \frac{4x^2}{r^2}\right)\frac{b^2x}{r^4} - C\frac{\beta^2-1}{\beta^3}\left[(2-\beta^2) - 2(5-\beta^2)\frac{x_2^2}{r_2^2}\right.$$
$$\left. + 8\frac{x_2^2}{r_2^2}\frac{x_2^2}{r_2^2} + \frac{\beta^2-1}{\beta}\left(3 - \frac{4x_2^2}{r_2^2}\right)\frac{bx_2}{r_2^2}\right]\frac{2b}{r_2^2} + \frac{(1-CD)}{(1-D)b}\left\{\sum_{n=1}^{\infty}\left[2\cos n\theta a_{-n}\left(\frac{b}{r}\right)^n\right]\right.$$
$$\left. - \sum_{n=1}^{\infty}\left[n\cos(n+2)\theta a_{-n}\left(\frac{b}{r}\right)^n\right] + \sum_{n=1}^{\infty}\left[(n+1)\cos(n+2)\theta a_{-n}\left(\frac{b}{r}\right)^{n+2}\right]\right\}$$
$$+ \frac{(1-CD)}{(1-C)b}\sum_{n=1}^{\infty}\left[\cos(n+2)\theta a_n\left(\frac{b}{r}\right)^{n+2}\right], \tag{A.4}$$

$$h_{xyx}^{(3)} = -2\left(1 - \frac{2x_1^2}{r_1^2}\right)\frac{x_1}{r_1^2} + \left[3C - D - 4C\frac{x_2^2}{r_2^2}\right]\frac{x_2}{r_2^2} - \left[3C - D - 4C\frac{x^2}{r^2}\right]\frac{x}{r^2} - \frac{1}{\beta}\left[1 - C - \beta\frac{1-CD}{1-C}a_0'\right]$$
$$\times\left(1 - \frac{2x^2}{r^2}\right)\frac{b}{r^2} + 2C\left(3 - \frac{4x^2}{r^2}\right)\frac{b^2x}{r^4} - C\frac{\beta^2-1}{\beta^3}\left[1 - \left(8 - \frac{8x_2^2}{r_2^2}\right)\frac{x_2^2}{r_2^2} + \frac{\beta^2-1}{\beta}\left(3 - \frac{4x_2^2}{r_2^2}\right)\frac{bx_2}{r_2^2}\right]\frac{2b}{r_2^2}$$
$$- \frac{(1-CD)}{(1-D)b}\left\{\sum_{n=1}^{\infty}\left[n\cos(n+2)\theta a_{-n}'\left(\frac{b}{r}\right)^n\right] + \sum_{n=1}^{\infty}\left[(n+1)\cos(n+2)\theta a_{-n}'\left(\frac{b}{r}\right)^{n+2}\right]\right\}$$
$$- \frac{(1-CD)}{(1-C)b}\sum_{n=1}^{\infty}\left[\cos(n+2)\theta a_n'\left(\frac{b}{r}\right)^{n+2}\right], \tag{A.5}$$

$$h_{xyy}^{(3)} = -2\left(1 - \frac{2x_1^2}{r_1^2}\right)\frac{y}{r_1^2} + \left[D + C\left(1 - \frac{4x_2^2}{r_2^2}\right)\right]\frac{y}{r_2^2} - \left[D + C\left(1 - \frac{4x^2}{r^2}\right)\right]\frac{y}{r^2}$$
$$- \frac{1}{\beta}\left[2C\beta^2 + \beta\frac{1-CD}{1-C}a_0 - C - 1\right]\frac{2bxy}{r^4} + 2C\left(1 - \frac{4x^2}{r^2}\right)\frac{b^2y}{r^4} + C\frac{\beta^2-1}{\beta^3}$$
$$\times\left[2\beta^2 - 4 + 8\frac{x_2^2}{r_2^2}\frac{x_2}{r_2^2} + \frac{\beta^2-1}{\beta}\left(1 - \frac{4x_2^2}{r_2^2}\right)\frac{b}{r_2^2}\right]\frac{2by}{r_2^2} + \frac{(1-CD)}{(1-D)b}\left\{\sum_{n=1}^{\infty}\left[n\sin(n+2)\theta a_{-n}\left(\frac{b}{r}\right)^n\right]\right.$$
$$\left. - \sum_{n=1}^{\infty}\left[(n+1)\sin(n+2)\theta a_{-n}\left(\frac{b}{r}\right)^{n+2}\right]\right\} - \frac{(1-CD)}{(1-C)b}\sum_{n=1}^{\infty}\left[\sin(n+2)\theta a_n\left(\frac{b}{r}\right)^{n+2}\right], \tag{A.6}$$

where

$$x_1 = x - e, \quad r_1 = x_1^2 + y^2, \quad x_2 = x - \frac{b^2}{e}, \quad r_2 = x_2^2 + y^2. \tag{A.7}$$

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